Section 6.6

Surfaces of revolution are the objects obtained by revolving a curve about an axis. Comparing this to solids of revolution, if a region R is bounded by a curve $y = f(x)$ and an axis $y = C$, the solid of revolution obtained from revolving R about $y = C$ has an outer surface that is a surface of revolution (and additional circular bases).

Frustums and our first surface area.

Consider the curve $y = mx$. The surface of revolution obtained from this curve over the interval $[0, b]$ is a cone sans base, where the radius of the base is $f(b)$, height is b and side-length is $l = \sqrt{b^2 + (f(b))^2} = \sqrt{b^2 + m^2b^2}$. The surface area is the fraction $\dfrac{2\pi r}{2\pi l} = \dfrac{r}{l}$ of the area of the circle with radius l. Hence

$$S_b = \pi f(b) l = \pi m b \sqrt{b^2 + m^2 b^2} = \pi m b^2 \sqrt{1 + m^2} \tag{6.1}$$

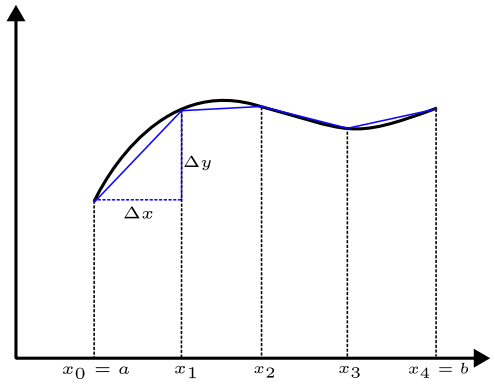
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A *frustum* is the surface obtained by revolving $y = mx$ over an interval $[a, b]$. By Equation 6.1, the surface area of the frustum is

$$\begin{aligned} S = S_b - S_a &= \pi m b^2 \sqrt{1 + m^2} - \pi m a^2 \sqrt{1 + m^2} \\ &= \pi m (b^2 - a^2) \sqrt{1 + m^2} \\ &= \pi m (b + a)(b - a) \sqrt{1 + m^2} \\ &= \pi (mb + ma) \sqrt{(b - a)^2 + (mb - ma)^2} \\ &= \pi (f(b) + f(a)) \sqrt{(b - a)^2 + (f(b) - f(a))^2} \end{aligned} \tag{6.2}$$

Surface Area Formula.

As with our calculation of arc length, we approximate a given curve $y = f(x)$ on $[a, b]$ by subdividing into n sub-intervals and forming the piece-wise linear function $P(x)$ equal to the line segment secant to $y = f(x)$ on each subinterval.



We then approximate the surface area of the surface of revolution obtained by revolving $y = f(x)$ about the x-axis via the area of the frustum associated to the linear pieces of $P(x)$. By Equation 6.2, this approximation is

$$\begin{aligned} S &\approx \sum_{i=1}^{n} \pi(f(x_{i-1}) + f(x_i))\sqrt{(\Delta x)^2 + (\Delta f_i)^2} \\ &= \sum_{i=1}^{n} \pi(f(x_{i-1}) + f(x_i))\sqrt{1 + \left(\frac{\Delta f_i}{\Delta x}\right)^2}\Delta x \\ &\approx \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + (f'(x_i^*)^2)}\Delta x \end{aligned}$$

This approximation has the form of a Riemann sum, improving as $n \to \infty$. Hence

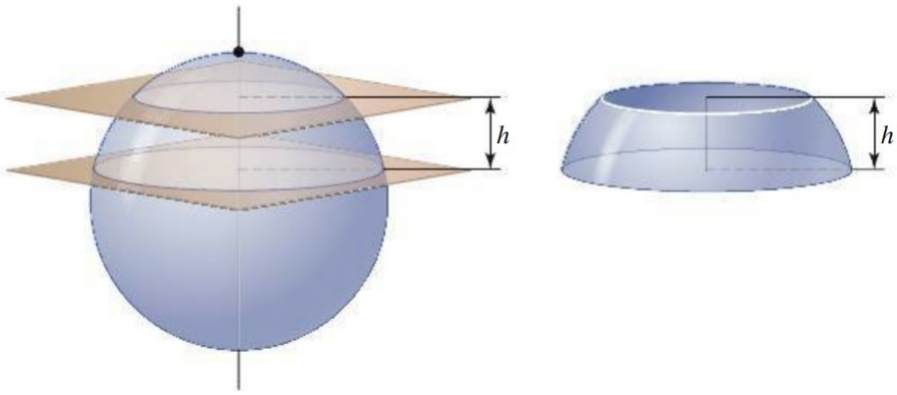
Definition 6.6.1. Let f be a nonnegative function with continuous first derivative on $[a, b]$. The area of the surface obtained when the graph of f on the interavl $[a, b]$ is revolved about the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}dx.$$

Example 6.6.2. Find the area of the surface generated by rotating the graph of $f(x) = 2\sqrt{x}$ on the interval $[1, 3]$ about the x-axis.

Example 6.6.3. Use the definition of surface area, Definition 6.6.1, to compute the surface area of a sphere of radius a by finding the surface area of the of the *spherical zone* obtained by rotating $y = \sqrt{a^2 - x^2}$ from $[0, h]$ about the x-axis.

Example 6.6.4. Show that the surface area of a spherical zone found between two parallel planes h units apart intersecting a sphere of radius a is always $S = 2\pi ah$ regardless of the location of the two planes.



Example 6.6.5. When the circle $(x-a)^2+y^2=r^2$ is revolved about the y-axis, the resulting surface is a torus when $|r|<a$. Show that the surface area of this torus in $S=4\pi^2 ar$.

